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A best constant for bivariate Bernstein and Szász-Mirakyan operators ☆

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Abstract

For classical Bernstein operators over the unit square, we obtain the best uniform constant in preservation of the usual l_∞-modulus of continuity, at the same time we show that it coincides with the corresponding best uniform constant for bivariate Szász operators. The result validates a conjecture stated in a previous paper. The proof involves both probabilistic and analytic arguments, as well as numerical computation of some specific values.

Keywords: Best constants; Modulus of continuity; Bernstein operators; Szász–Mirakyan operators; Bivariate operators; Binomial distribution; Poisson distribution

1. Introduction and main result

For $n, k = 1, 2, \ldots$ let $B_n^{\langle k \rangle} := B_n \otimes \cdots \otimes B_n$ be the tensor product of k copies of the classical Bernstein operator over the interval $[0, 1]$ given by

$$B_n f(x) := \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k},$$

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and denote by $C_n^{\langle k \rangle}(\delta)$ the best constant in preservation of the usual modulus of continuity for the l_∞-norm in $\mathbb{R}^k$, that is

$$C_n^{\langle k \rangle}(\delta) := \sup_{f \in \mathscr{F}_k} \frac{\omega(B_n^{\langle k \rangle} f; \delta)}{\omega(f; \delta)}, \quad 0 < \delta \leqslant 1,$$

where $\mathscr{F}_k$ is the set of all real non-constant bounded functions on $[0,1]^k$, and $\omega(f; \cdot)$ stands for the above-mentioned modulus of continuity, i.e.,

$$\omega(f; \delta) := \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in [0,1]^k, \ ||\mathbf{x} - \mathbf{y}||_\infty \leqslant \delta\}, \quad \delta \geqslant 0.$$

On the other hand, for $t > 0$ and $k = 1, 2, \ldots$, let $S_t^{\langle k \rangle} := S_t \otimes \cdots \otimes S_t$ be the tensor product of k copies of the Szász–Mirakyan operator S_t over the interval $[0, \infty)$ given by

$$S_t g(x) := \sum_{k=0}^{\infty} g(k/t) e^{-tx} \frac{(tx)^k}{k!},$$

and denote by $D_t^{\langle k \rangle}(\delta)$ the corresponding best constant for such an operator, i.e.,

$$D_t^{\langle k \rangle}(\delta) := \sup_{g \in \mathscr{G}_k} \frac{\omega(S_t^{\langle k \rangle} g; \delta)}{\omega(g; \delta)}, \quad \delta > 0,$$

where $\mathscr{G}_k$ is the set of all real non-constant functions g on $[0, \infty)^k$ such that $\omega(g; 1) < \infty$ (or, equivalently, $\omega(g; \delta) < \infty$ for all $\delta > 0$).

Many facts about $C_n^{\langle k \rangle}(\delta)$ and $D_t^{\langle k \rangle}(\delta)$ are known in the literature. In [4], the reader can find explicit probabilistic formulae for such constants (to be used in the next section) which generalize the one-dimensional formulae given in [1]. From the formula for $D_t^{\langle k \rangle}(\delta)$, it follows that

$$D_t^{\langle k \rangle}(\delta) = D_1^{\langle k \rangle}(t\delta), \quad t, \delta > 0,$$

implying that the best uniform constant $\sup_{\delta > 0} D_t^{\langle k \rangle}(\delta)$ only depends upon the dimension k. The facts

$$\sup_{0 < \delta \leqslant 1} C_n^{\langle 1 \rangle}(\delta) = 2 \quad (n \geqslant 1) \quad \text{and} \quad \sup_{\delta > 0} D_t^{\langle 1 \rangle}(\delta) = 2 - e^{-1} \quad (t > 0)$$

were respectively established in [2,6]. It was shown in [5] that, for $k \geqslant 3$,

$$\sup_{0 < \delta \leqslant 1} C_n^{\langle k \rangle}(\delta) = k = \sup_{\delta > 0} D_1^{\langle k \rangle}(\delta), \quad n \geqslant 1,$$

that is, the best uniform constants coincide with the dimension, and (as in the one-dimensional case) the one for $B_n^{\langle k \rangle}$ does not depend upon the parameter n, while, in the case $k = 2$, the value of $\sup_{0 < \delta \leqslant 1} C_n^{\langle 2 \rangle}(\delta)$ *depends* upon n, and both the values of $\sup_{n \geqslant 1} \sup_{0 < \delta \leqslant 1} C_n^{\langle 2 \rangle}(\delta)$ and $\sup_{\delta > 0} D_1^{\langle 2 \rangle}(\delta)$ lie in the interval $[2, 5/2]$. As for the exact value of these quantities, on the basis of certain computational evidence, it was conjectured the following.

Theorem. *We have*

$$\sup_{\delta>0} D_1^{\langle 2\rangle}(\delta) = \sup_{n\geqslant 1} \sup_{0<\delta\leqslant 1} C_n^{\langle 2\rangle}(\delta)$$
$$= 1 - e^{-2} + \sum_{j=0}^{\infty}\left[1 - e^{-2}\left(\sum_{i=0}^{j}\frac{1}{i!}\right)^2\right] = 2.3884423\ldots \quad . \tag{1}$$

In the present paper, we give a theoretical proof of this result.

2. Auxiliary results

In this section, we introduce some notations, restate the preceding theorem in a more convenient form for our purposes, and collect some necessary auxiliary results.

We set, for $n\geqslant 1$ and $0<x\leqslant n$,

$$C_n(x) := C_n^{\langle 2\rangle}(x/n).$$

We recall that, according to the formulae in [3], we have

$$C_n(x) = E\left\lceil\frac{\eta_n(x)}{x}\right\rceil,$$

where E denotes mathematical expectation, $\lceil\cdot\rceil$ is the ceiling function, and $\eta_n(x) := \eta_n'(x)\vee\eta_n''(x)$ is the maximum of two independent integer-valued random variables $\eta_n'(x)$ and $\eta_n''(x)$ having the same binomial distribution given by

$$P(\eta_n'(x)=k) = p_{n,k}(x) := \begin{cases} \binom{n}{k}\left(\frac{x}{n}\right)^k\left(1-\frac{x}{n}\right)^{n-k} & k=0,1,\ldots,n,\\ 0 & \text{otherwise.}\end{cases}$$

We also denote by

$$C_n^*(x) := P(\eta_n(x)>0) + \frac{E\eta_n(x)}{x} = 1 - p_{n,0}^2(x) + \frac{E\eta_n(x)}{x},$$

and recall that

$$E\eta_n(x) = \sum_{k=1}^{\infty} kP(\eta_n(x)=k) = \sum_{k=1}^{\infty} P(\eta_n(x)\geqslant k)$$
$$= \sum_{k=0}^{n-1}\left[1-\left(\sum_{j=0}^{k} p_{n,j}(x)\right)^2\right].$$

Similarly, we set, for $x>0$,

$$D(x) := D_1^{\langle 2\rangle}(x) = E\left\lceil\frac{\xi(x)}{x}\right\rceil,$$

and

$$D^*(x) := P(\xi(x) > 0) + \frac{E\xi(x)}{x}$$
$$= 1 - \pi_0^2(x) + \frac{1}{x}\sum_{k=0}^{\infty}\left[1 - \left(\sum_{j=0}^{k}\pi_j(x)\right)^2\right],$$

where $\xi(x) := \xi'(x) \vee \xi''(x)$ is the maximum of two independent random variables $\xi'(x)$ and $\xi''(x)$ having the same Poisson distribution of parameter x, i.e.,

$$P(\xi'(x) = k) = \pi_k(x) := e^{-x}\frac{x^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

With the preceding notations, it is clear that (1) can be rewritten as follows:

$$\sup_{n \geqslant 1} \sup_{0 < x \leqslant n} C_n(x) = \sup_{x > 0} D(x) = D^*(1). \tag{2}$$

The point is that the functions $C_n(\cdot)$ and $D(\cdot)$ are quite irregular and hardly tractable, but $C_n^*(\cdot)$ and $D^*(\cdot)$ are fairly smooth. The following lemmas collect the necessary facts for the proof of (2) given in the next section.

Lemma 1. *We have*

(a)

$$C_n(x) \leqslant C_n^*(x), \quad n \geqslant 1, \quad 0 < x \leqslant n.$$

(b)

$$D(x) \leqslant D^*(x), \quad x > 0.$$

Proof. Both inequalities are nothing but particular cases of the inequality

$$E\lceil U \rceil \leqslant P(U > 0) + EU,$$

which holds true for every nonnegative random variable U. □

Lemma 2. *We have*

$$C_n^*(x) \leqslant D^*(x), \quad n \geqslant 1, \ 0 < x \leqslant n.$$

Proof. Fix $n \geqslant 1$ and $x \in (0, n]$, and denote by

$$a_{n,k} := P(\eta_n'(x) \leqslant k) = \sum_{j=0}^{k} p_{n,j}(x),$$

$$b_k := P(\xi'(x) \leqslant k) = \sum_{j=0}^{k} \pi_j(x),$$

for $k = 0, 1, 2, \ldots$. From some results of Anderson and Samuels [3, Corollary 2.1], there is an integer $r \geqslant 1$ such that

$$\begin{aligned} &a_{n,k} \leqslant b_k, \quad 0 \leqslant k \leqslant r-1, \\ &a_{n,k} \geqslant b_k, \quad k \geqslant r. \end{aligned} \tag{3}$$

We have

$$\begin{aligned} E\xi(x) - E\eta_n(x) &= \sum_{k=0}^{\infty} (1 - b_k^2) - \sum_{k=0}^{\infty} (1 - a_{n,k}^2) \\ &= \sum_{k=0}^{\infty} (a_{n,k}^2 - b_k^2) \\ &= a_{n,0}^2 - b_0^2 + \sum_{k=1}^{\infty} (a_{n,k} + b_k)(a_{n,k} - b_k) \\ &\geqslant a_{n,0}^2 - b_0^2 + (a_{n,r} + b_r) \sum_{k=1}^{\infty} (a_{n,k} - b_k), \end{aligned}$$

the inequality by (3) and the fact that the sequence $\{a_{n,k} + b_k : k \geqslant 0\}$ is nondecreasing. Since

$$\begin{aligned} \sum_{k=1}^{\infty} (a_{n,k} - b_k) &= \left[\sum_{k=0}^{\infty} (1 - b_k) - \sum_{k=0}^{\infty} (1 - a_{n,k}) \right] - (a_{n,0} - b_0) \\ &= [E\xi'(x) - E\eta_n{}'(x)] - (a_{n,0} - b_0) \\ &= -(a_{n,0} - b_0) \end{aligned}$$

(the last equality because $E\xi'(x) = x = E\eta_n{}'(x)$), and

$$a_{n,r} + b_r \geqslant a_{n,1} + b_1 \geqslant (1 + x)(a_{n,0} + b_0),$$

we finally obtain that

$$E\xi(x) - E\eta_n(x) \geqslant x(b_0^2 - a_{n,0}^2),$$

which is another way to express the conclusion. □

Lemma 3. *We have*:

(a)

$$x \frac{d}{dx} E\eta_n(x) = E\eta_n(x) - \sum_{k=1}^{n} k p_{n,k}^2(x), \quad n \geqslant 1, \ x \in (0, n].$$

(b)

$$x \frac{d}{dx} E\xi(x) = E\xi(x) - \sum_{k=1}^{\infty} k \pi_k^2(x), \quad x > 0.$$

Proof. Let $n \geqslant 1$ be fixed. It is immediate that we have, for $x \in (0, n]$ and $k \geqslant 0$,

$$x \frac{d}{dx} p_{n,k}(x) = k p_{n,k}(x) - (k+1) p_{n,k+1}(x),$$

implying that

$$x\frac{d}{dx}\sum_{j=0}^{k} p_{n,j}(x) = -(k+1)p_{n,k+1}(x),$$

and, therefore,

$$\begin{aligned}
x\frac{d}{dx}\left(\sum_{j=0}^{k} p_{n,j}(x)\right)^2 &= -(k+1)2P(\eta_n'(x)\leqslant k)P(\eta_n''(x)=k+1)\\
&= -(k+1)[P(\eta_n'(x)\leqslant k)P(\eta_n''(x)=k+1)\\
&\quad + P(\eta_n''(x)\leqslant k)P(\eta_n'(x)=k+1)]\\
&= -(k+1)[P(\eta_n(x)=k+1)-p_{n,k+1}^2(x)].
\end{aligned}$$

We conclude that

$$\begin{aligned}
x\frac{d}{dx}E\eta_n(x) &= x\frac{d}{dx}\sum_{k=0}^{n-1}\left[1-\left(\sum_{j=0}^{k} p_{n,j}(x)\right)^2\right]\\
&= \sum_{k=1}^{\infty} kP(\eta_n(x)=k) - \sum_{k=1}^{n} kp_{n,k}^2(x)\\
&= E\eta_n(x) - \sum_{k=1}^{n} kp_{n,k}^2(x),
\end{aligned}$$

showing part (a). The proof of (b) is achieved in the same way, by starting from the fact that we have

$$x\frac{d}{dx}\pi_k(x) = k\pi_k(x) - (k+1)\pi_{k+1}(x),$$

for all $x>0$ and $k\geqslant 0$. $\square$

Lemma 4. (a) *For each* $n\geqslant 1$, *the function* $x^{-1}E\eta_n(x)$ *is decreasing in* $(0,n]$.

(b) *The function* $x^{-1}E\xi(x)$ *is decreasing in* $(0,\infty)$.

(c) *The function* $D^*(\cdot)$ *is increasing in* $(0,1]$ *and decreasing in* $[3/2,\infty)$.

Proof. From the preceding lemma, we have

$$\frac{d}{dx}\frac{E\eta_n(x)}{x} = -\frac{1}{x^2}\sum_{k=1}^{n} kp_{n,k}^2(x) < 0, \quad n\geqslant 1,\ x\in(0,n],$$

and

$$\frac{d}{dx}\frac{E\xi(x)}{x} = -\frac{1}{x^2}\sum_{k=1}^{\infty} k\pi_k^2(x) = -\sum_{k=0}^{\infty}\frac{1}{k+1}\pi_k^2(x) < 0, \quad x>0, \tag{4}$$

showing parts (a) and (b). From (4), we also have, for $x > 0$,

$$\begin{aligned}\frac{d}{dx}D^*(x) &= \frac{d}{dx}\left[1 - \pi_0^2(x) + \frac{E\xi(x)}{x}\right]\\ &= 2\pi_0^2(x) - \sum_{k=0}^{\infty} \frac{1}{k+1}\pi_k^2(x)\\ &= \pi_0^2(x) - \sum_{k=1}^{\infty} \frac{1}{k+1}\pi_k^2(x)\\ &= e^{-2x}\left[1 - \sum_{k=1}^{\infty} \frac{1}{k+1}\left(\frac{x^k}{k!}\right)^2\right].\end{aligned}$$

If $x \in (0, 1]$, we have

$$\sum_{k=1}^{\infty} \frac{1}{k+1}\left(\frac{x^k}{k!}\right)^2 \leqslant \sum_{k=1}^{\infty} \frac{1}{k+1}\left(\frac{1}{k!}\right)^2 < \frac{1}{2}(e-1) < 1,$$

implying that $\frac{d}{dx}D^*(x) > 0$, while, for $x \geqslant 3/2$,

$$\sum_{k=1}^{\infty} \frac{1}{k+1}\left(\frac{x^k}{k!}\right)^2 \geqslant \sum_{k=1}^{\infty} \frac{1}{k+1}\left(\frac{(3/2)^k}{k!}\right)^2 > \frac{1}{2}\left(\frac{3}{2}\right)^2 > 1,$$

which implies that $\frac{d}{dx}D^*(x) < 0$. This shows part (c), and completes the proof of the lemma. □

Lemma 5. *We have*:

(a) $\lim_{x\uparrow 1} C_n(x) = C_n^*(1)$, $n \geqslant 1$.
(b) $\lim_{n\to\infty} C_n^*(1) = D^*(1)$.
(c) $\lim_{x\uparrow 1} D(x) = D^*(1)$.
(d) $E\xi(1) = D^*(1) - 1 + e^{-2} = 1.52377761\ldots$.
(e) $D^*(1.55) = 2.38835554\ldots$.

Proof. We have, for $n \geqslant 1$,

$$\begin{aligned}\lim_{x\uparrow 1} C_n(x) &= \lim_{x\uparrow 1} \sum_{k=1}^{n} \left\lceil \frac{k}{x} \right\rceil P(\eta_n(x) = k) = \sum_{k=1}^{n} (k+1)P(\eta_n(1) = k)\\ &= P(\eta_n(1) > 0) + E\eta_n(1) = C_n^*(1),\end{aligned}$$

showing part (a). Part (c) is shown in the same way, and we omit the details. Part (b) readily follows from the fact that

$$\lim_{n\to\infty} p_{n,k}(1) = \pi_k(1), \quad k = 0, 1, 2, \ldots$$

(i.e., the Poisson approximation to the binomial distribution). Finally, parts (d) and (e) merely are numerical computations. □

3. Proof of the theorem

Recall the numerical value of $D^*(1)$ appearing in (1). We have, by Lemmas 1, 2, 4(c) and 5(e),

$$D(x) \leqslant D^*(x) \leqslant D^*(1), \quad 0 < x \leqslant 1,$$

$$C_n(x) \leqslant C_n^*(x) \leqslant D^*(x) \leqslant D^*(1), \quad 0 < x \leqslant 1, \; n \geqslant 1,$$

$$D(x) \leqslant D^*(x) \leqslant D^*(1.55) < D^*(1), \quad x \geqslant 1.55,$$

$$C_n(x) \leqslant C_n^*(x) \leqslant D^*(x) \leqslant D^*(1.55) < D^*(1), \quad n \geqslant 2, \; 1.55 \leqslant x \leqslant n.$$

Let $1 < x < 1.55$. Using the fact that the random variable $\xi(x)$ is integer-valued, and Lemmas 4(b) and 5(d), we obtain

$$\begin{aligned} D(x) &\leqslant E\lceil \xi(x) \rceil = E\xi(x) \leqslant xE\xi(1) \\ &\leqslant 1.55E\xi(1) = 2.361855\ldots < D^*(1), \end{aligned}$$

and, analogously, by Lemmas 2 and 4(a),

$$C_n(x) \leqslant E\lceil \eta_n(x) \rceil = E\eta_n(x) \leqslant xE\eta_n(1) \leqslant x\, E\xi(1) < D^*(1), \quad n \geqslant 2.$$

From all the above, we conclude that

$$\sup_{n \geqslant 1} \sup_{0 < x \leqslant n} C_n(x) \leqslant D^*(1) \quad \text{and} \quad \sup_{x > 0} D(x) \leqslant D^*(1). \tag{5}$$

Finally, we have, by Lemma 5(c)

$$\sup_{x > 0} D(x) \geqslant \lim_{x \uparrow 1} D(x) = D^*(1),$$

and, by Lemma 5(a,b),

$$\sup_{n \geqslant 1} \sup_{0 < x \leqslant n} C_n(x) \geqslant \lim_{n \to \infty} \lim_{x \uparrow 1} C_n(x) = D^*(1),$$

showing that the inequalities in (5) actually are equalities, and finishing the proof of (2).

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